DERIVATION AND ANALYSIS OF A NEW $2D$ GREEN-NAGHDI SYSTEM

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ABSTRACT. We derive here a variant of the $2D$ Green-Naghdi equations that model the propagation of two-directional, nonlinear dispersive waves in shallow water. This new model has the same accuracy as the standard $2D$ Green-Naghdi equations. Its mathematical interest is that it allows a control of the rotational part of the (vertically averaged) horizontal velocity, which is not the case for the usual Green-Naghdi equations. Using this property, we show that the solution of these new equations can be constructed by a standard Picard iterative scheme so that there is no loss of regularity of the solution with respect to the initial condition. Finally, we prove that the new Green-Naghdi equations conserve the almost irrotationality of the vertically averaged horizontal component of the velocity.

1. Introduction

1.1. **General setting.** The water-waves problem, consists in studying the motion of the free surface and the evolution of the velocity field of a layer of fluid under the following assumptions: the fluid is ideal, incompressible, irrotationnal, and under the only influence of gravity. Many works have set a good theoretical background for this problem. Its local well-posedness has been discussed among others by Nalimov [13], Yosihara [18], Craig [5], Wu [15, 16] and Lannes [8]. Since we are interested here in the asymptotic behavior of the solutions, it is convenient to work with a non-dimensionalized version of the equations. In this framework, (see for instance [2] for details), the free surface is parametrized by $z = \varepsilon\zeta(t, X)$ (with $X = (x, y) \in \mathbb{R}^2$) and the bottom by $z = -1 + \beta b(X)$. Here, ε and β are dimensionless parameters defined as

$$\varepsilon = \frac{a_{surf}}{h_0}, \quad \beta = \frac{b_{bott}}{h_0};$$

where a_{surf} is the typical amplitude of the waves and b_{bott} is the typical amplitude of the bottom deformations, while h_0 is the depth. One can use the incompressibility and irrotationality conditions to write the non-dimensionalized water-waves equations under Bernoulli's formulation, in terms of a velocity potential φ associated to the flow, where $\varphi(t, .)$ is defined on $\Omega_t = \{(X, z), -1+\beta b(x) < z < \varepsilon\zeta(t, X)\}$ (i.e. the velocity field is given by $U = \nabla_{X,z}\varphi$) :

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$$
\left\{
\begin{array}{lll}
\mu\partial_x^2\varphi+\mu\partial_y^2\varphi+\partial_z^2\varphi=0, & \text{at} & -1+\beta b<z<\varepsilon\zeta,\\
\partial_z\varphi-\mu\beta\nabla b\cdot\nabla\varphi=0 & \text{at} & z=-1+\beta b,\\
\partial_t\zeta-\dfrac{1}{\mu}(\mu\varepsilon\nabla\zeta\cdot\nabla\varphi+\partial_z\varphi)=0, & \text{at} & z=\varepsilon\zeta,\\
\partial_t\varphi+\dfrac{1}{2}(\varepsilon|\nabla\varphi|^2+\dfrac{\varepsilon}{\mu}(\partial_z\varphi)^2)+\zeta=0 & \text{at} & z=\varepsilon\zeta,
\end{array}
\right.
\tag{1}
$$

where $\nabla=(\partial_x,\partial_y)^T=(\partial_1,\partial_2)^T$. The *shallowness* parameter μ appearing in this set of equations is defined as

$$\mu=\frac{h_0^2}{\lambda^2},$$

where, λ is the typical wave-length of the waves. Making assumptions on the size of ε, β, and μ one is led to derive (simpler) asymptotic models from (1). In the shallow-water scaling ($\mu\ll 1$), one can derive (when no smallness assumption is made on ε and β) the standard Green-Naghdi equations (see §2.1 below for a derivation and [2] for a rigorous justification). For two-dimensional surfaces and over uneven bottoms these equations couple the free surface elevation ζ to the vertically averaged horizontal component of the velocity,

$$v(t,X)=\frac{1}{1+\varepsilon\zeta-\beta b}\int_{-1+\beta b}^{\varepsilon\zeta}\nabla\varphi(t,X,z)dz, \tag{2}$$

and can be written as:

$$
\left\{
\begin{array}{l}
\partial_t\zeta+\nabla\cdot(hv)=0,\\
\big(h+\mu\mathcal{T}[h,\beta b]\big)\partial_t v+h\nabla\zeta+\varepsilon\big(h+\mu\mathcal{T}[h,\beta b]\big)(v\cdot\nabla)v\\
\qquad+\mu\varepsilon\big\{\dfrac{2}{3}\nabla[(h^3(\partial_1 v\cdot\partial_2 v^\perp+(\nabla\cdot v)^2)]+\Re[h,\beta b](v)\big\}=0,
\end{array}
\right.
\tag{3}
$$

where $h=1+\varepsilon\zeta-\beta b$, $v=(V_1,V_2)^T$, $v^\perp=(-V_2,V_1)^\perp$, and

$$
\begin{aligned}
\mathcal{T}[h,\varepsilon b]W \;=\;& -\frac{1}{3}\nabla(h^3\nabla\cdot W)+\frac{\beta}{2}[\nabla(h^2\nabla b\cdot W)-h^2\nabla b\nabla\cdot W]\\
&+\beta^2 h\nabla b\nabla b\cdot W,
\end{aligned}
\tag{4}
$$

while the purely topographic term $\Re[h,\beta b](v)$ is defined as:

$$
\begin{aligned}
\Re[h,\beta b](v) \;=\;& \frac{\beta}{2}\nabla\big(h^2(V_1^2\partial_1^2 b+2V_1V_2\partial_1\partial_2 b+V_2^2\partial_2^2 b)\big)\\
&+\beta h^2\big(\partial_1 v\cdot\partial_2 v^\perp+(\nabla\cdot v)^2\big)\nabla b\\
&+\beta^2 h\big(V_1^2\partial_1^2 b+2V_1V_2\partial_1\partial_2 b+V_2^2\partial_2^2 b\big)\nabla b.
\end{aligned}
\tag{5}
$$

A rigorous justification of the standard GN model was given by Li [11] in $1D$ and for flat bottoms, and by B. Alvarez-Samaniego and D. Lannes [2] in the general case. This latter reference relies on well-posedness results for these equations given in [3] and based on general well-posedness results for evolution equations using a Nash-Moser scheme. The result of [3] covers both the case of $1D$ and $2D$ surfaces, and allows for non flat bottoms. The reason why a Nash-Moser scheme is used there is because the estimates on the linearized equations exhibit losses of derivatives. However, in the $1D$ case, such losses do not occur and it is possible to construct a solution with a standard Picard iterative scheme as in [11, 7] with flat and non flat bottoms respectively. But, this is not the case in $2D$, since for instance, the term

$\partial_1 v \cdot \partial_2 v^\perp$ is not controled by the energy norm $|\cdot|_{Y^s}$ naturally associated to (3) (see [3]),

$$|(\zeta,v)|^2_{Y^s} = |\zeta|^2_{H^s} + |v|^2_{H^s} + \mu|\nabla\cdot v|^2_{H^s}.$$

This is the motivation for the present derivation of a new variant of the $2D$ Green-Naghdi equations (3). This variant has the same accuracy as the standard $2D$ Green-Naghdi equations (3) (see §2.2 below for a derivation) and can be written under the form:

$$\left\{\begin{array}{l} \partial_t\zeta + \nabla\cdot(hv) = 0, \\ \Big(h+\mu\big(\mathcal{T}[h,\beta b] - \nabla^\perp \mathrm{curl}\ \big)\Big)\partial_t v + h\nabla\zeta \\ \quad + \varepsilon\Big(h+\mu\big(\mathcal{T}[h,\beta b] - \nabla^\perp \mathrm{curl}\ \big)\Big)(v\cdot\nabla)v \\ \quad + \mu\varepsilon\Big\{\dfrac{2}{3}\nabla[h^3(\partial_1 v\cdot\partial_2 v^\perp + (\nabla\cdot v)^2)] + \Re[h,\beta b](v)\Big\} = 0, \end{array}\right. \tag{6}$$

where $\nabla^\perp = (-\partial_y,\partial_x)^T$, $\mathrm{curl}\, v = \partial_1 V_2 - \partial_2 V_1$, while the linear operators $\mathcal{T}[h,\varepsilon b]$ and $\Re[h,\beta b]$ are defined in (4) and (5).

The reason why the new terms (involving $\nabla^\perp \mathrm{curl}$) do not affect the precision of the model is because the solutions to (3) are nearly irrotational (in the sense that $\mathrm{curl}\, v$ is small). This property is of course satisfied also by our new model (6). The presence of these new terms allows the definition of a new energy norm that controls also the rotational part of v. Consequently, we show that it is possible to use a standard Picard iterative scheme to prove the well-posedness of (6), so that there is no loss of regularity of the solution with respect to the initial condition.

1.2. **Organization of the paper.** We first recall the derivation of the standard $2D$ Green-Naghdi equations in Section 2.1 while in Section 2.2 we derive the new model (6). We give some preliminary results in Section 3.1; the main theorem which proves the well-posedness of this new Green-Naghdi system is then stated in Section 3.2 and proved in Section 3.3. Finally, in Section 3.4 we prove that (6) conserves the almost irrotationality of v.

1.3. **Notation.** We denote by $C(\lambda_1,\lambda_2,...)$ a constant depending on the parameters λ_1, λ_2, ... and *whose dependence on the λ_j is always assumed to be nondecreasing.* The notation $a \lesssim b$ means that $a \le Cb$, for some nonegative constant C whose exact expression is of no importance (*in particular, it is independent of the small parameters involved*).

Let p be any constant with $1 \le p < \infty$ and denote $L^p = L^p(\mathbb{R}^2)$ the space of all Lebesgue-measurable functions f with the standard norm

$$|f|_{L^p} = \Big(\int_{\mathbb{R}^2} |f(X)|^p dX\Big)^{1/p} < \infty.$$

When $p=2$, we denote the norm $|\cdot|_{L^2}$ simply by $|\cdot|_2$. The inner product of any functions f_1 and f_2 in the Hilbert space $L^2(\mathbb{R}^2)$ is denoted by

$$(f_1,f_2) = \int_{\mathbb{R}^2} f_1(X) f_2(X) dX.$$

The space $L^\infty = L^\infty(\mathbb{R}^2)$ consists of all essentially bounded, Lebesgue-measurable functions f with the norm

$$|f|_{L^\infty} = \operatorname{ess\,sup} |f(X)| < \infty.$$

We denote by $W^{1,\infty} = W^{1,\infty}(\mathbb{R}^2) = \{f \in L^\infty, \nabla f \in (L^\infty)^2\}$ endowed with its canonical norm.
For any real constant s, $H^s = H^s(\mathbb{R}^2)$ denotes the Sobolev space of all tempered distributions f with the norm $|f|_{H^s} = |\Lambda^s f|_2 < \infty$, where Λ is the pseudo-differential operator $\Lambda = (1-\Delta)^{1/2}$.
For any functions $u = u(x,t)$ and $v(x,t)$ defined on $\mathbb{R}^2 \times [0,T)$ with $T > 0$, we denote the inner product, the L^p-norm and especially the L^2-norm, as well as the Sobolev norm, with respect to the spatial variable X, by $(u,v) = (u(\cdot,t), v(\cdot,t))$, $|u|_{L^p} = |u(\cdot,t)|_{L^p}$, $|u|_{L^2} = |u(\cdot,t)|_{L^2}$, and $|u|_{H^s} = |u(\cdot,t)|_{H^s}$, respectively.
Let $C^k(\mathbb{R}^2)$ denote the space of k-times continuously differentiable functions and $C_0^\infty(\mathbb{R}^2)$ denote the space of infinitely differentiable functions, with compact support in $\mathbb{R}^2$; we also denote by $C_b^\infty(\mathbb{R}^2)$ the space of infinitely differentiable functions that are bounded together with all their derivatives.
For any closed operator T defined on a Banach space X of functions, the commutator $[T,f]$ is defined by $[T,f]g = T(fg) - fT(g)$ with f, g and fg belonging to the domain of T.
We denote $v^\perp = (-V_2, V_1)^T$ and $\operatorname{curl} v = \partial_1 V_2 - \partial_2 V_1$ where $v = (V_1, V_2)^T$.

2. Derivation of the new Green-Naghdi model

This section is devoted to the derivation of a new Green-Naghdi asymptotic model for the water-waves equations in the shallow water ($\mu \ll 1$) of the same accuracy as the standard $2D$ Green-Naghdi equations (3).

2.1. **Derivation of the standard Green-Naghdi equations (3).**

We recall here the main steps of [10] for the derivation of the standard $2D$ Green-Naghdi equations (3). In order to reduce the model (1) into a model of two equations we introduce the trace of the velocity potential at the free surface, defined as

$$\psi = \varphi_{|_{z=\varepsilon\zeta}},$$

and the Dirichlet-Neumann operator $\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\cdot$ as

$$\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\mu\varepsilon\nabla\zeta \cdot \nabla\varphi_{|_{z=\varepsilon\zeta}} + \partial_z\varphi_{|_{z=\varepsilon\zeta}},$$

with φ solving the boundary value problem

$$\begin{cases} \mu\partial_x^2\varphi + \mu\partial_y^2\varphi + \partial_z^2\varphi = 0, \\ \partial_n\varphi_{|_{z=-1+\beta b}} = 0, \\ \varphi_{|_{z=\varepsilon\zeta}} = \psi. \end{cases} \tag{7}$$

As remarked in [19, 6], the equations (1) are equivalent to a set of two equations on the free surface parametrization ζ and the trace of the velocity potential at the surface $\psi = \varphi_{|_{z=\varepsilon\zeta}}$ involving the Dirichlet-Neumann operator. Namely

$$\begin{cases} \partial_t\zeta + \dfrac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = 0, \\ \partial_t\psi + \zeta + \dfrac{\varepsilon}{2}|\nabla\psi|^2 - \varepsilon\mu\dfrac{(\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi + \nabla(\varepsilon\zeta)\cdot\nabla\psi)^2}{2(1+\varepsilon^2\mu|\nabla\zeta|^2)} = 0. \end{cases} \tag{8}$$

It is a straightforward consequence of Green's identity that

$$\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi = -\nabla\cdot(hv),$$

with $h = 1 + \varepsilon\zeta - \beta b$ and $v = \frac{1}{1+\varepsilon\zeta-\beta b}\int_{-1+\beta b}^{\varepsilon\zeta} \nabla\varphi(t, X, z)dz$. Therefore, the first equation of (8) exactly coincides with the first equation of (3). In order to derive the evolution equation on v, the key point is to obtain an asymptotic expansion of $\nabla\psi$ with respect to μ and in terms pf v and ζ. As in [10], we look for an asymptotic expansion of φ under the form

$$\varphi_{app} = \sum_{j=0}^{N} \mu^j \varphi_j. \tag{9}$$

Plugging this expression into the boundary value problem (7) one can cancel the residual up to the order $O(\mu^{N+1})$ provided that

$$\forall j = 0, ..., N, \qquad \partial_z^2 \varphi_j = -\partial_x^2 \varphi_{j-1} - \partial_y^2 \varphi_{j-1} \tag{10}$$

(with the convention that $\varphi_{-1} = 0$), together with the boundary conditions

$$\forall j = 0, ..., N, \qquad \begin{cases} \varphi_{j_{|z=\varepsilon\zeta}} = \delta_{0,j}\psi, \\ \partial_z \varphi_j = \beta\nabla b \cdot \nabla\varphi_{j-1_{|z=-1+\beta b}} \end{cases} \tag{11}$$

(where $\delta_{0,j} = 1$ if $j = 0$ and 0 otherwise).
By solving the ODE (10) with the boundary conditions (11), one finds (see [10])

$$\varphi_0 = \psi, \tag{12}$$

$$\varphi_1 = (z - \varepsilon\zeta)\big(-\frac{1}{2}(z + \varepsilon\zeta) - 1 + \beta b\big)\Delta\psi + \beta(z - \varepsilon\zeta)\nabla b.\nabla\psi. \tag{13}$$

According to formulae (12), (13), the horizontal component of the velocity in the fluid domain is given by

$$V(z) = \nabla\varphi_0(z) + \nabla\varphi_1(z) + O(\mu^2).$$

The averaged velocity is thus given by

$$v = \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta} (\nabla\varphi_0(z) + \nabla\varphi_1(z))\, dz + O(\mu^2),$$

or equivalently

$$v = \nabla\psi - \mu\frac{1}{h}\mathcal{T}[h, \beta b]\nabla\psi + O(\mu^2),$$

and thus

$$\nabla\psi = v + \mu\frac{1}{h}\mathcal{T}[h, \beta b]\nabla\psi + O(\mu^2), \tag{14}$$

where $\mathcal{T}[h, \beta b]$ is as in (4). As in [10], taking the gradient of the second equation of (8), replacing $\nabla\psi$ by its expression (14) and $\frac{1}{\mu}\mathcal{G}_\mu[\varepsilon\zeta, \beta b]\psi$ by $-\nabla \cdot (hv)$ in the resulting equation, gives the standard Green-Naghdi equations (after dropping the $O(\mu^2)$ terms),

$$\begin{cases} \partial_t\zeta + \nabla \cdot (hv) = 0, \\ \big(h + \mu\mathcal{T}[h, \beta b]\big)\partial_t v + h\nabla\zeta + \varepsilon\big(h + \mu\mathcal{T}[h, \beta b]\big)(v \cdot \nabla)v \\ \qquad + \mu\varepsilon\big\{\frac{2}{3}\nabla[h^3(\partial_1 v \cdot \partial_2 v^\perp + (\nabla \cdot v)^2)] + \Re[h, \beta b](v)\big\} = 0, \end{cases}$$

where $h = 1 + \varepsilon\zeta - \beta b$, $v = (V_1, V_2)^T$, $v^\perp = (-V_2, V_1)^T$, and the linear operators $\mathcal{T}[h, \varepsilon b]$ and $\Re[h, \beta b]$ being defined in (4) and (5).

2.2. **Derivation of the new Green-Naghdi system (6).** It is quite common in the literature to work with variants of the Green-Naghdi equations (6) that differ only up to terms of order $O(\mu^2)$ in order to improve the dispersive properties of the model (see for instance [14, 4]) or to change its mathematical properties [12]. Our approach here is in the same spirit since our goal is to derive a new model with better energy estimates.
In order to obtain the new Green-Naghdi system (6), let us remark that, from the expression of v one has

$$\nabla\psi = v + \frac{\mu}{h}\mathcal{T}[h,\beta b]\nabla\psi + O(\mu^2);$$

and since $v = \nabla\psi + O(\mu)$, one gets the following Lemma:

Lemma 1. *Let v be the vertically averaged horizontal component of the velocity given above, one obtains*

$$\begin{aligned} \operatorname{curl}\partial_t v &= O(\mu), \\ \operatorname{curl}(v\cdot\nabla)v &= O(\mu). \end{aligned} \tag{15}$$

Remark 1. *For the sake of simplicity, we denote by $O(\mu)$ any family of functions $(f_\mu)_{0<\mu<1}$ such that $\big(\frac{1}{\mu}f_\mu\big)_{0<\mu<1}$ remains bounded in $L^\infty([0,\frac{T}{\varepsilon}],H^r(\mathbb{R}^2))$, for some r large enough.*

Proof. By applying the operator $(\operatorname{curl}\partial_t)(\cdot)$ to the identity $v = \nabla\psi + O(\mu)$ one gets

$$\operatorname{curl}\partial_t v = O(\mu).$$

In order to prove the second identity of (15), replace $v = \nabla\psi + O(\mu)$ in $(v\cdot\nabla)v$ and apply the operator $\operatorname{curl}(\cdot)$ to deduce

$$\operatorname{curl}(v\cdot\nabla)v = O(\mu).$$

$\square$

Using Lemma 1, the quantities $\mu\nabla^\perp\operatorname{curl}\partial_t v$ and $\mu\nabla^\perp\operatorname{curl}\varepsilon(v\cdot\nabla)v$ are of size $O(\mu^2)$, which is the precision of the GN equations (3). We can thus include these new terms in the second equation of (3) to get

$$\begin{cases} \partial_t\zeta + \nabla\cdot(hv) = 0, \\ \Big(h + \mu\big(\mathcal{T}[h,\beta b] - \nabla^\perp\operatorname{curl}\big)\Big)\partial_t v + h\nabla\zeta \\ \quad + \varepsilon\Big(h + \mu\big(\mathcal{T}[h,\beta b] - \nabla^\perp\operatorname{curl}\big)\Big)(v\cdot\nabla)v \\ \quad + \mu\varepsilon\Big\{\frac{2}{3}\nabla[h^3(\partial_1 v\cdot\partial_2 v^\perp + (\nabla\cdot v)^2)] + \Re[h,\beta b](v)\Big\} = O(\mu^2), \end{cases}$$

(the linear operators $\mathcal{T}[h,\varepsilon b]$ and $\Re[h,\beta b]$ being defined in (4) and (5)).

Remark 2. *We added the quantity $\mu\nabla^\perp\operatorname{curl}\partial_t v = O(\mu^2)$ to the standard GN equations (3) to obtain an energy norm $|\cdot|_{X^s}$:*

$$|(\zeta,v)|^2_{X^s} = |\zeta|^2_{H^s} + |v|^2_{H^s} + \mu|\nabla\cdot v|^2_{H^s} + \mu|\operatorname{curl} v|^2_{H^s}.$$

The last term is absent from the energy $|\cdot|_{Y^s}$ associated to the standard GN equations (3) (see [3]). We will see in the next section that it allows a control of the term $\partial_1 v\cdot\partial_2 v^\perp$.

Remark 3. *The bilinear operator* $\Re[h,\beta b](v)$ *only involves second order derivatives of* v *while third order derivatives of* v *have been factorized by* $h+\mu\big(\mathcal{T}[h,\varepsilon b]-\nabla^{\perp}curl\big)$. *The fact that the operator* $\Re[h,\beta b](v)$ *does not involve third order derivatives is of great interest for the well-posedness of the new Green-Naghdi model.*

3. Mathematical analysis of the new Green-Naghdi model

3.1. **Preliminary results.** For the sake of simplicity, we take here and throughout the rest of this paper ($\beta=\varepsilon$) and we write

$$\mathfrak{T}=h+\mu\big(\mathcal{T}[h,\varepsilon b]-\nabla^{\perp}\mathrm{curl}\,\big).$$

We always assume that the nonzero depth condition

$$\exists\, h_{min}>0,\quad \inf_{X\in\mathbb{R}^2} h\geq h_{min},\quad h=1+\varepsilon(\zeta-b) \tag{16}$$

is valid initially, which is a necessary condition for the new GN type system (6) to be physically valid. We shall demonstrate that the operator $\mathfrak{T}$ plays an important role in the energy estimate and the local well-posedness of the GN type system (6). Therefore, we give here some of its properties.

The following lemma gives an important invertibility result on $\mathfrak{T}$ and some properties of the inverse operator $\mathfrak{T}^{-1}$.

Lemma 2. *Let* $b\in C_b^{\infty}(\mathbb{R}^2)$, $t_0>1$ *and* $\zeta\in H^{t_0+1}(\mathbb{R}^2)$ *be such that (16) is satisfied. Then, the operator* $\mathfrak{T}$ *has a bounded inverse on* $(L^2(\mathbb{R}^2))^2$, *and*
(i) For all $0\leq s\leq t_0+1$, *one has*

$$|\mathfrak{T}^{-1}f|_{H^s}+\sqrt{\mu}|\nabla\cdot\mathfrak{T}^{-1}f|_{H^s}+\sqrt{\mu}|\mathrm{curl}\,\mathfrak{T}^{-1}f|_{H^s}\leq C(\frac{1}{h_{min}},|h-1|_{H^{t_0+1}})|f|_{H^s},$$

and

$$\sqrt{\mu}|\mathfrak{T}^{-1}\nabla g|_{H^s}+\sqrt{\mu}|\mathfrak{T}^{-1}\nabla^{\perp}g|_{H^s}\leq C(\frac{1}{h_{min}},|h-1|_{H^{t_0+1}})|g|_{H^s}.$$

(ii) If $s\geq t_0+1$ *and* $\zeta\in H^s(\mathbb{R}^2)$ *then*

$$\|\,\mathfrak{T}^{-1}\,\|_{(H^s)^2\to(H^s)^2}+\sqrt{\mu}\,\|\,\mathfrak{T}^{-1}\nabla\,\|_{(H^s)^2\to(H^s)^2}+\sqrt{\mu}\,\|\,\mathfrak{T}^{-1}\nabla^{\perp}\,\|_{(H^s)^2\to(H^s)^2}\leq c_s,$$

and

$$\sqrt{\mu}\,\|\,\nabla\cdot\mathfrak{T}^{-1}\,\|_{(H^s)^2\to(H^s)^2}+\sqrt{\mu}\,\|\,\mathrm{curl}\,\mathfrak{T}^{-1}\,\|_{(H^s)^2\to(H^s)^2}\leq c_s,$$

where c_s *is a constant depending on* $\frac{1}{h_{min}}$, $|h-1|_{H^s}$ *and independent of* $(\mu,\varepsilon)\in(0,1)^2$.

Remark 4. *Here and throughout the rest of this paper, and for the sake of simplicity, we do not try to give some optimal regularity assumption on the bottom parametrization* b. *This could easily be done, but is of no interest for our present purpose. Consequently, we omit to write the dependence on* b *of the different quantities that appear in the proof.*

Proof. It can be remarked that the operator $\mathfrak{T}$ is L^2 self-adjoint; since, one has

$$\begin{aligned}&\Big(h+\mu\big(\mathcal{T}[h,\varepsilon b]-\nabla^{\perp}\mathrm{curl}\,\big)v,v\Big)=(hv,v)\\ &+\mu\Big(h\Big(\frac{h}{\sqrt{3}}\nabla\cdot v-\varepsilon\frac{\sqrt{3}}{2}\nabla b\cdot v\Big),\frac{h}{\sqrt{3}}\nabla\cdot v-\varepsilon\frac{\sqrt{3}}{2}\nabla b\cdot v\Big)+\frac{\mu\varepsilon^2}{4}(h\nabla b\cdot v,\nabla b\cdot v)\\ &+\mu(\mathrm{curl}\,v,\mathrm{curl}\,v),\end{aligned}$$

and using the fact that $\inf_{\mathbb{R}^2} h \geq h_{min}$, one deduces that

$$(\mathfrak{T}v, v) \geq E(\varepsilon b, v),$$

with

$$E(\varepsilon b, v) := h_{min}|v|_2^2 + \mu h_{min}|\frac{h}{\sqrt{3}}\nabla\cdot v - \varepsilon\frac{\sqrt{3}}{2}\nabla b\cdot v|_2^2 + \frac{\mu\varepsilon^2 h_{min}}{4}|\nabla b\cdot v|_2^2 + \mu|\mathrm{curl}\, v|_2^2,$$

proceeding exactly as in the proof of the Lemma 1 of [7] it follows that $\mathfrak{T}$ has an inverse bounded on $(L^2(\mathbb{R}^2))^2$.
For the rest of the proof, One can proceeding as in the proof of Lemma 4.7 of [3], to get the result. $\square$

3.2. **Linear analysis of (6).** In order to rewrite the new GN type equations (6) in a condensed form, let us write $U = (\zeta, v^T)^T$, $v = (V_1, V_2)^T$ and

$$Q = h^3(-\partial_1 v\cdot\partial_2 v^{\perp} - (\nabla\cdot v)^2).$$

We decompose now the $O(\varepsilon\mu)$ term of the second equation of (6) under the form

$$-\frac{2}{3}\nabla Q + R[h, \varepsilon b](v) = R_1[U]v + r_2(U),$$

with for all $f = (F_1, F_2)^T$

$$\begin{aligned} R_1[U]f &= -\frac{2}{3}\nabla Q[U]f + \frac{\varepsilon}{2}\nabla(h^2(V_1F_1\partial_1^2 b + 2V_1F_2\partial_1\partial_2 b + V_2F_2\partial_2^2 b)) \\ &\quad -\varepsilon h^2(-\partial_1 v\cdot\partial_2 f^{\perp} - (\nabla\cdot v)(\nabla\cdot f))\nabla b; \\ r_2(U) &= \varepsilon^2 h(V_1^2\partial_1^2 b + 2V_1V_2\partial_1\partial_2 b + V_2^2\partial_2^2 b)\nabla b, \end{aligned}$$

where

$$Q[U]f = h^3(-\partial_1 v\cdot\partial_2 f^{\perp} - (\nabla\cdot v)(\nabla\cdot f)),$$

(in particular, $Q = Q[U]v$). The new Green-Naghdi equations (6) can thus be written after applying $\mathfrak{T}^{-1}$ to both sides of the second equation in (6) as

$$\partial_t U + A[U]U + B(U) = 0, \tag{17}$$

with $U = (\zeta, V_1, V_2)^T$, $v = (V_1, V_2)^T$ and where

$$A[U] = \begin{pmatrix} \varepsilon v\cdot\nabla & h\nabla\cdot \\ \mathfrak{T}^{-1}(h\nabla) & \varepsilon(v\cdot\nabla) + \varepsilon\mu\mathfrak{T}^{-1}R_1[U] \end{pmatrix} \tag{18}$$

and

$$B(U) = \begin{pmatrix} \varepsilon\nabla b\cdot v \\ \varepsilon\mu\mathfrak{T}^{-1}r_2(U) \end{pmatrix}. \tag{19}$$

This subsection is devoted to the proof of energy estimates for the following initial value problem around some reference state $\underline{U} = (\underline{\zeta}, \underline{V_1}, \underline{V_2})^T$:

$$\begin{cases} \partial_t U + A[\underline{U}]U + B(\underline{U}) = 0; \\ U_{|t=0} = U_0. \end{cases} \tag{20}$$

We define first the X^s spaces, which are the energy spaces for this problem.

Definition 1. *For all $s \geq 0$ and $T > 0$, we denote by X^s the vector space $H^s(\mathbb{R}^2) \times (H^{s+1}(\mathbb{R}^2))^2$ endowed with the norm*

$$\forall\, U = (\zeta, v) \in X^s, \quad |U|^2_{X^s} := |\zeta|^2_{H^s} + |v|^2_{(H^s)^2} + \mu|\nabla \cdot v|^2_{H^s} + \mu|\mathrm{curl}\, v|^2_{H^s},$$

while X^s_T stands for $C([0, \frac{T}{\varepsilon}]; X^s)$ endowed with its canonical norm.

We define the matrix S as

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \underline{\mathfrak{T}} \end{pmatrix}, \tag{21}$$

with $\underline{h} = 1 + \varepsilon(\underline{\zeta} - b)$ and $\underline{\mathfrak{T}} = \underline{h} + \mu(\mathcal{T}[\underline{h}, \varepsilon b] - \nabla^\perp(\nabla\wedge\ ))$. A natural energy for the IVP (20) is given by

$$E^s(U)^2 = (\Lambda^s U, S\Lambda^s U). \tag{22}$$

The link between $E^s(U)$ and the X^s-norm is investigated in the following Lemma.

Lemma 3. *Let $b \in C^\infty_b(\mathbb{R}^2)$, $s \geq 0$ and $\underline{\zeta} \in W^{1,\infty}(\mathbb{R}^2)$. Under the condition (16), $E^s(U)$ is uniformly equivalent to the $|\cdot|_{X^s}$-norm with respect to $(\mu, \varepsilon) \in (0,1)^2$:*

$$E^s(U) \leq C\big(|\underline{h}|_{W^{1,\infty}}\big)|U|_{X^s},$$

and

$$|U|_{X^s} \leq C\big(\frac{1}{h_{min}}\big)E^s(U).$$

Proof. Notice first that

$$E^s(U)^2 = |\Lambda^s\zeta|^2_2 + (\Lambda^s v, \underline{\mathfrak{T}}\Lambda^s v),$$

one gets the first estimate using the explicit expression of $\underline{\mathfrak{T}}$, integration by parts and Cauchy-Schwarz inequality.
The other inequality can be proved by using that $\inf_{x\in\mathbb{R}^2} h \geq h_{min} > 0$ and proceeding as in the proof of Lemma 2. □

We prove now the energy estimates in the following proposition. It is worth insisting on the fact that these estimates are uniform with respect to $\varepsilon, \mu \in (0,1)$; since the control of the $s+1$ order derivatives by the X^s-norm disappears as $\mu \to 0$, the uniformity with respect to μ requires particular care (see the control of B_{46} in the proof for instance), but is very important for the application since $\mu \ll 1$.

Proposition 1. *Let $b \in C^\infty_b(\mathbb{R}^2)$, $t_0 > 1$, $s \geq t_0 + 1$. Let also $\underline{U} = (\underline{\zeta}, \underline{V_1}, \underline{V_2})^T \in X^s_T$ be such that $\partial_t \underline{U} \in X^{s-1}_T$ and satisfying the condition (16) on $[0, \frac{T}{\varepsilon}]$. Then for all $U_0 \in X^s$ there exists a unique solution $U = (\zeta, V_1, V_2)^T \in X^s_T$ to (20) and for all $0 \leq t \leq \frac{T}{\varepsilon}$*

$$E^s(U(t)) \leq e^{\varepsilon\lambda_T t}E^s(U_0) + \varepsilon\int_0^t e^{\varepsilon\lambda_T(t-t')}C(E^s(\underline{U})(t'))dt',$$

for some $\lambda_T = \lambda_T(\sup_{0\leq t\leq T/\varepsilon} E^s(\underline{U}(t)), \sup_{0\leq t\leq T/\varepsilon} |\partial_t \underline{h}(t)|_{L^\infty})$.

Proof. Existence and uniqueness of a solution to the IVP (20) is achieved by using classical methods as in appendix A of [7] for the standard $1D$ Green-Naghdi equations and we thus focus our attention on the proof of the energy estimate. For any $\lambda \in \mathbb{R}$, we compute

$$e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(U)^2) = -\varepsilon\lambda E^s(U)^2 + \partial_t(E^s(U)^2).$$

Since
$$E^s(U)^2=(\Lambda^s U, S\Lambda^s U),$$
and $U=(\zeta,V_1,V_2)^T$, $v=(V_1,V_2)^T$, we have
$$\partial_t(E^s(U)^2)=2(\Lambda^s\zeta,\Lambda^s\zeta_t)+2(\Lambda^s v,\underline{\mathfrak{T}}\Lambda^s v_t)+(\Lambda^s v,[\partial_t,\underline{\mathfrak{T}}]\Lambda^s v). \tag{23}$$
One gets using the equations (20) and integrating by parts,
$$\begin{aligned}\frac{1}{2}e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(U)^2)=&-\frac{\varepsilon\lambda}{2}E^s(U)^2-(SA[\underline{U}]\Lambda^s U,\Lambda^s U)\\ &-\big([\Lambda^s,A[\underline{U}]]U,S\Lambda^s U\big)-(\Lambda^s B(\underline{U}),S\Lambda^s U)+\frac{1}{2}(\Lambda^s v,[\partial_t,\underline{\mathfrak{T}}]\Lambda^s v).\end{aligned}\tag{24}$$
We now turn to bound from above the different components of the r.h.s of (24).

• Estimate of $(SA[\underline{U}]\Lambda^s U,\Lambda^s U)$. Remarking that
$$SA[\underline{U}]=\begin{pmatrix}\varepsilon\underline{v}\cdot\nabla & \underline{h}\nabla\cdot\\ \underline{h}\nabla & \underline{\mathfrak{T}}(\varepsilon\underline{v}\cdot\nabla)+\varepsilon\mu R_1[\underline{U}]\end{pmatrix},$$
we get
$$\begin{aligned}(SA[\underline{U}]\Lambda^s U,\Lambda^s U)&=(\varepsilon\underline{v}\cdot\nabla\Lambda^s\zeta,\Lambda^s\zeta)+(\underline{h}\nabla\cdot\Lambda^s v,\Lambda^s\zeta)\\ &\quad+(\underline{h}\nabla\Lambda^s\zeta,\Lambda^s v)+\big((\underline{\mathfrak{T}}(\varepsilon\underline{v}\cdot\nabla)+\varepsilon\mu R_1[\underline{U}])\Lambda^s v,\Lambda^s v\big)\\ &=: A_1+A_2+A_3+A_4.\end{aligned}$$
We now focus on the control of $(A_j)_{1\le j\le 4}$.

− Control of A_1. Integrating by parts, one obtains
$$A_1=(\varepsilon\underline{v}\cdot\nabla\Lambda^s\zeta,\Lambda^s\zeta)=-\frac{1}{2}(\varepsilon\nabla\cdot\underline{v}\Lambda^s\zeta,\Lambda^s\zeta),$$
and one can conclude by Cauchy-Schwarz inequality that
$$|A_1|\le\varepsilon C(|\nabla\cdot\underline{v}|_{L^\infty})E^s(U)^2.$$
− Control of A_2+A_3. First remark that
$$|A_2+A_3|=|(\nabla\underline{h}\cdot\Lambda^s v,\Lambda^s\zeta)|\le|\nabla\underline{h}|_{(L^\infty)^2}E^s(U)^2;$$
we get,
$$|A_2+A_3|\le\varepsilon C(|\nabla\underline{h}|_{(L^\infty)^2}E^s(U)^2.$$
− Control of A_4. One computes,
$$\begin{aligned}A_4&=\varepsilon\big(\underline{\mathfrak{T}}((\underline{v}\cdot\nabla)\Lambda^s v),\Lambda^s v\big)+(\varepsilon\mu R_1[\underline{U}]\Lambda^s v,\Lambda^s v)\\ &=: A_{41}+A_{42}.\end{aligned}$$
Note first that
$$\begin{aligned}A_{41}&=\varepsilon(\underline{h}\ (\underline{v}\cdot\nabla)\Lambda^s v,\Lambda^s v)+\frac{\varepsilon\mu}{3}(\underline{h}^3\ \nabla\cdot(\underline{v}\cdot\nabla)\Lambda^s v),\Lambda^s\nabla\cdot v)\\ &\quad-\frac{\varepsilon^2\mu}{2}(\underline{h}^2\ \nabla b\cdot(\underline{v}\cdot\nabla)\Lambda^s v,\Lambda^s\nabla\cdot v)-\frac{\varepsilon^2\mu}{2}(\underline{h}^2\nabla b\nabla\cdot(\underline{v}\cdot\nabla)\Lambda^s v,\Lambda^s v)\\ &\quad+\varepsilon^3\mu(\underline{h}\ \nabla b\nabla b\cdot(\underline{v}\cdot\nabla)\Lambda^s v,\Lambda^s v)+\varepsilon\mu(\mathrm{curl}\,(\underline{v}\cdot\nabla)\Lambda^s v),\Lambda^s\mathrm{curl}\,v);\end{aligned}$$
remark also that
$$\begin{aligned}(\mathrm{curl}\,(\underline{v}\cdot\nabla)\Lambda^s v),\Lambda^s\mathrm{curl}\,v)&=-\frac{1}{2}\Big((\mathrm{curl}\,\Lambda^s v,\partial_1\underline{V_1}\mathrm{curl}\,\Lambda^s v)+(\nabla\wedge\Lambda^s v,\partial_2\underline{V_2}\nabla\wedge\Lambda^s v)\Big)\\ &\quad+(\mathrm{curl}\,\Lambda^s v,\partial_1\underline{v}\cdot\nabla\Lambda^s V_2)-(\mathrm{curl}\,\Lambda^s v,\partial_2\underline{v}\cdot\nabla\Lambda^s V_1),\end{aligned}$$

and that, for all F and G smooth enough, one has

$$((G\cdot\nabla)v, F) = -(v, F\,\nabla\cdot G) - (v, (G\cdot\nabla)F).$$

By using successively the above identities, the following relation (25)

$$|\nabla F_1|_2^2 + |\nabla F_2|_2^2 = |\nabla\cdot F|_2^2 + |\mathrm{curl}\, F|_2^2, \tag{25}$$

integration by parts and the Cauchy-Schwarz inequality, one obtains directly:

$$|A_{41}| \leq \varepsilon C(|\underline{v}|_{(W^{1,\infty})^2}, |\underline{\zeta}|_{W^{1,\infty}}) E^s(U)^2.$$

For A_{42}, remark that

$$\begin{aligned}
|A_{42}| &= |(\varepsilon\mu R_1[\underline{U}]\Lambda^s v, \Lambda^s v)|\\
&= \Big| + \frac{2}{3}\varepsilon\mu(Q[\underline{U}]\Lambda^s v, \Lambda^s\nabla\cdot v) - \mu\varepsilon^2\Big(\underline{h}^2(-\partial_1\underline{v}\cdot\partial_2\Lambda^s v^\perp - (\nabla\cdot\underline{v})(\nabla\cdot\Lambda^s v))\nabla b, \Lambda^s v\Big)\\
&\quad -\frac{\mu\varepsilon^2}{2}\Big(\underline{h}^2(\underline{V}_1\Lambda^s V_1\partial_1^2 b + 2\underline{V}_1\Lambda^s V_2\partial_1\partial_2 b + \underline{V}_2\Lambda^s V_2\partial_2^2 b), \Lambda^s\nabla\cdot v\Big)\Big|,
\end{aligned}$$

where

$$Q[\underline{U}]f = \underline{h}^3(-\partial_1\underline{v}\cdot\partial_2 f^\perp - (\nabla\cdot\underline{v})(\nabla\cdot f)).$$

we deduce that

$$|A_{42}| \leq \varepsilon C(|\underline{v}|_{(W^{1,\infty})^2}, |\underline{\zeta}|_{W^{1,\infty}}) E^s(U)^2.$$

The estimates proved in A_{41} and A_{42} show that

$$|A_4| \leq \varepsilon C(|\underline{v}|_{(W^{1,\infty})^2}, |\underline{\zeta}|_{W^{1,\infty}}) E^s(U)^2.$$

• Estimate of $\big([\Lambda^s, A[\underline{U}]]U, S\Lambda^s U\big)$. Remark first that

$$\begin{aligned}
\big([\Lambda^s, A[\underline{U}]]U, S\Lambda^s U\big) &= ([\Lambda^s, \varepsilon\underline{v}]\cdot\nabla\zeta, \Lambda^s\zeta) + ([\Lambda^s, \underline{h}]\nabla\cdot v, \Lambda^s\zeta)\\
&+ ([\Lambda^s, \underline{\mathfrak{T}}^{-1}\,\underline{h}]\nabla\zeta, \underline{\mathfrak{T}}\Lambda^s v) + ([\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \underline{\mathfrak{T}}\Lambda^s v) + \varepsilon\mu\big([\Lambda^s, \underline{\mathfrak{T}}^{-1}R_1[\underline{U}]]v, \underline{\mathfrak{T}}\Lambda^s v\big)\\
&=: B_1 + B_2 + B_3 + B_4 + B_5.
\end{aligned}$$

− Control of $B_1 + B_2 = ([\Lambda^s, \varepsilon\underline{v}]\cdot\nabla\zeta, \Lambda^s\zeta) + ([\Lambda^s, \underline{h}]\nabla\cdot v, \Lambda^s\zeta)$.
Since $s \geq t_0 + 1$, we can use the following commutator estimate (26) (see e.g [9])

$$|[\Lambda^s, F]G|_2 \lesssim |\nabla F|_{H^{s-1}}|G|_{H^{s-1}}. \tag{26}$$

to get

$$|B_1 + B_2| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

− Control of $B_4 = ([\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \underline{\mathfrak{T}}\Lambda^s v)$. By using the explicit expression of $\underline{\mathfrak{T}}$ we get

$$\begin{aligned}
B_4 &= ([\Lambda^s, \varepsilon\underline{v}\cdot\nabla]v, \underline{h}\Lambda^s v) + \frac{\mu}{3}(\nabla\cdot[\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \underline{h}^3\,\Lambda^s\nabla\cdot v)\\
&\quad -\frac{\varepsilon\mu}{2}([\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \underline{h}^2\,\nabla b\Lambda^s\nabla\cdot v) + \frac{\varepsilon\mu}{2}([\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \nabla(\underline{h}^2\,\nabla b\cdot\Lambda^s v))\\
&\quad +\varepsilon^2\mu([\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \underline{h}\,\nabla b\nabla b\cdot\Lambda^s v) + \mu(\mathrm{curl}\,[\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \Lambda^s\mathrm{curl}\, v)\\
&:= B_{41} + B_{42} + B_{43} + B_{44} + B_{45} + B_{46}.
\end{aligned}$$

One obtains as for the control of B_1 and B_2 above that

$$|B_{4j}| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2, \quad j\in\{1,3,4,5\}.$$

The control of B_{42} and B_{46} is more delicate because of the dependence on μ (recall that the energy $E^s(U)$ controls $s+1$ derivatives of v, but with a small coefficient

$\sqrt{\mu}$ in front of the derivatives of order $O(\sqrt{\mu})$). For B_{46}, we thus proceed as follows: we first write

$$\begin{aligned}
B_{46} &= \mu(\operatorname{curl}[\Lambda^s, \varepsilon(\underline{v}\cdot\nabla)]v, \Lambda^s \operatorname{curl} v)\\
&= \mu(\partial_1[\Lambda^s, \varepsilon\underline{V_1}\partial_1]V_2, \Lambda^s\operatorname{curl} v) + \mu(\partial_1[\Lambda^s, \varepsilon\underline{V_2}\partial_2]V_2, \Lambda^s\operatorname{curl} v)\\
&\quad -\mu(\partial_2[\Lambda^s, \varepsilon\underline{V_1}\partial_1]V_1, \Lambda^s\operatorname{curl} v) - \mu(\partial_2[\Lambda^s, \varepsilon\underline{V_2}\partial_2]V_1, \Lambda^s\operatorname{curl} v)\\
&:= B_{461} + B_{462} + B_{463} + B_{464}.
\end{aligned}$$

Remarking that for all $j \in \{1,2\}$ we have

$$\partial_j[\Lambda^s, f]g = [\Lambda^s, \partial_j f]g + [\Lambda^s, f]\partial_j g,$$

and

$$[\Lambda^s, f\partial_j]g = [\Lambda^s, f]\partial_j g,$$

we can rewrite B_{461} under the form

$$\begin{aligned}
B_{461} &= \mu(\partial_1[\Lambda^s, \varepsilon\underline{V_1}\partial_1]V_2, \Lambda^s\operatorname{curl} v)\\
&= \mu([\Lambda^s, \varepsilon\partial_1\underline{V_1}]\partial_1 V_2, \Lambda^s\operatorname{curl} v) + \mu([\Lambda^s, \varepsilon\underline{V_1}]\partial_1^2 V_2, \Lambda^s\operatorname{curl} v).
\end{aligned}$$

It is then easy to use the commutator estimate (26) in order to obtain

$$|B_{46j}| \le \varepsilon C(E^s(\underline{U}))E^s(U)^2, \quad j \in \{1,2,3,4\}.$$

Similarly, B_{42} is controled by $\varepsilon C(E^s(\underline{U}))E^s(U)^2$. The estimates proved in $(B_{4j})_{j\in\{1,2,3,4,5,6\}}$ show that

$$|B_4| \le \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

− Control of $B_3 = ([\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\nabla\zeta, \underline{\mathfrak{T}}\Lambda^s v)$. Remark first that

$$\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}]\underline{h}\nabla\zeta = \underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\nabla\zeta - [\Lambda^s, \underline{h}]\nabla\zeta;$$

moreover, since $[\Lambda^s, \underline{\mathfrak{T}}^{-1}] = -\underline{\mathfrak{T}}^{-1}[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}$, one gets

$$\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\nabla\zeta = -[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta + [\Lambda^s, \underline{h}]\nabla\zeta,$$

and one can check by using the explicit expression of $\underline{\mathfrak{T}}$ that

$$\begin{aligned}
\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\nabla\zeta &= -[\Lambda^s, \underline{h}]\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta + \frac{\mu}{3}\nabla\{[\Lambda^s, \underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta)\}\\
&\quad -\frac{\varepsilon\mu}{2}\nabla[\Lambda^s, \underline{h}^2\nabla b]\cdot\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta + \frac{\varepsilon\mu}{2}[\Lambda^s, \underline{h}^2\nabla b]\nabla\cdot(\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta)\\
&\quad -\varepsilon^2\mu[\Lambda^s, \underline{h}\nabla b\nabla b^T]\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta + [\Lambda^s, \underline{h}]\nabla\zeta.
\end{aligned}$$

Integrating by parts yields

$$\begin{aligned}
B_3 &= \big(\underline{\mathfrak{T}}[\Lambda^s, \underline{\mathfrak{T}}^{-1}\ \underline{h}]\nabla\zeta, \Lambda^s v\big)\\
&= -\big([\Lambda^s, \underline{h}]\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta, \Lambda^s v\big) - \frac{\mu}{3}\big(\{[\Lambda^s, \underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta)\}, \Lambda^s\nabla\cdot v\big)\\
&\quad +\frac{\varepsilon\mu}{2}\big([\Lambda^s, \underline{h}^2\nabla b]\cdot\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta, \Lambda^s\nabla\cdot v\big) + \frac{\varepsilon\mu}{2}\big([\Lambda^s, \underline{h}^2\nabla b]\nabla\cdot(\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta), \Lambda^s v\big)\\
&\quad -\varepsilon^2\mu\big([\Lambda^s, \underline{h}\nabla b\nabla b^T]\underline{\mathfrak{T}}^{-1}\underline{h}\nabla\zeta, \Lambda^s v\big) + \big([\Lambda^s, \underline{h}]\nabla\zeta, \Lambda^s v\big).
\end{aligned}$$

One deduces directly from Lemma 2, the commutator estimate (26), and Cauchy-Schwarz inequality that

$$\begin{aligned}|B_3| \quad \leq \quad & C\big(\frac{1}{h_{min}}, |\underline{h}-1|_{H^s}\big)\Big\{\Big(\big(|\nabla\underline{h}|_{H^{s-1}}+\varepsilon^2\mu|\underline{h}^2\nabla b\nabla b^T|_{H^s}+\frac{\varepsilon\mu}{2}|\underline{h}\nabla b|_{H^s}\big)|\underline{h}\nabla\zeta|_{H^{s-1}} \\ & +\big(\frac{1}{3}|\nabla\underline{h}^3|_{H^{s-1}}+\frac{\varepsilon\sqrt{\mu}}{2}|\underline{h}^2\nabla b|_{H^s}\big)|\underline{h}\nabla\zeta|_{H^{s-1}}+|\nabla\underline{h}|_{H^{s-1}}|\nabla\zeta|_{H^{s-1}}\Big)\Big\}|v|_{X^s}.\end{aligned}$$

Finally, we deduce

$$|B_3| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

$-$ Control of $B_5 = \varepsilon\mu\big(\big[\Lambda^s, \underline{\mathfrak{T}}^{-1}R_1[\underline{U}]\big]v, \underline{\mathfrak{T}}\Lambda^s v\big)$. Let us first write

$$\underline{\mathfrak{T}}\big[\Lambda^s, \underline{\mathfrak{T}}^{-1}R_1[\underline{U}]\big]v = -[\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v + \big[\Lambda^s, R_1[\underline{U}]\big]v$$

so, that

$$\begin{aligned}\underline{\mathfrak{T}}\big[\Lambda^s, \underline{\mathfrak{T}}^{-1}R_1[\underline{U}]\big]v \quad = \quad & -[\Lambda^s, \underline{h}]\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v + \frac{\mu}{3}\nabla\{[\Lambda^s, \underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v)\} \\ & -\frac{\varepsilon\mu}{2}\nabla\{[\Lambda^s, \underline{h}^2\nabla b]\cdot\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v\} + \frac{\varepsilon\mu}{2}[\Lambda^s, \underline{h}^2\nabla b]\nabla\cdot(\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v) \\ & -\varepsilon^2\mu[\Lambda^s, \underline{h}\nabla b\nabla b^T]\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v + [\Lambda^s, R_1[\underline{U}]]v.\end{aligned}$$

To control the term $\big(\big[\Lambda^s, R_1[\underline{U}]\big]v, \Lambda^s v\big)$ we use the explicit expression of $R_1[\underline{U}]$:

$$\begin{aligned}R_1[\underline{U}]f \quad = \quad & -\frac{2}{3}\nabla Q[\underline{U}]f + \frac{\varepsilon}{2}\nabla(\underline{h}^2(\underline{V_1}F_1\partial_1^2 b + 2\underline{V_1}F_2\partial_1\partial_2 b + \underline{V_2}F_2\partial_2^2 b)) \\ & -\varepsilon h^2(-\partial_1\underline{v}\cdot\partial_2 f^{\perp} - (\nabla\cdot\underline{v})(\nabla\cdot f))\nabla b,\end{aligned}$$

where,

$$Q[\underline{U}]f = \underline{h}^3(-\partial_1\underline{v}\partial_2 f^{\perp} - (\nabla\cdot\underline{v})(\nabla\cdot f)).$$

As for the control of the term $\big(\nabla\{[\Lambda^s, \underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v)\}, \Lambda^s v\big)$ we use the explicit expression of $R_1[\underline{U}]$, the relation (25), the commutator estimate (26) and Lemma 2. Indeed,

$$\begin{aligned}\big(\nabla\{[\Lambda^s, \underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}R_1[\underline{U}]v)\}, \Lambda^s v\big) &= -\frac{2}{3}\big([\Lambda^s, \underline{h}^3]\nabla\cdot(\underline{\mathfrak{T}}^{-1}\nabla Q[\underline{U}]v), \Lambda^s\nabla\cdot v\big) \\ - \quad & \frac{\varepsilon}{2}\big([\Lambda^s, \underline{h}^3]\nabla\cdot\underline{\mathfrak{T}}^{-1}\nabla(\underline{h}^2(\underline{V_1}V_1\partial_1^2 b + 2\underline{V_1}V_2\partial_1\partial_2 b + \underline{V_2}V_2\partial_2^2 b)), \Lambda^s\nabla\cdot v\big) \\ - \quad & \varepsilon\big([\Lambda^s, \underline{h}^3]\nabla\cdot\underline{\mathfrak{T}}^{-1}(h^2(-\partial_1\underline{v}\cdot\partial_2\Lambda^s v^{\perp} - (\nabla\cdot\underline{v})(\nabla\cdot v))\nabla b), \Lambda^s\nabla\cdot v\big).\end{aligned}$$

and thus, after remarking that

$$|\nabla\cdot(\underline{\mathfrak{T}}^{-1}\nabla Q[\underline{U}]v)|_{H^{s-1}} \quad \leq \quad |\underline{\mathfrak{T}}^{-1}\nabla Q[\underline{U}]v|_{H^s},$$

we can proceed as for the control of B_3 to get

$$|B_5| \leq \varepsilon C(E^s(\underline{U}))E^s(U)^2.$$

• Estimate of $(\Lambda^s B(\underline{U}), S\Lambda^s U)$. Note first that

$$B(\underline{U}) = \begin{pmatrix} \varepsilon\nabla b\cdot\underline{v} \\ \varepsilon\mu\underline{\mathfrak{T}}^{-1}r_2(\underline{U}) \end{pmatrix}$$

so that

$$\begin{aligned}(\Lambda^s B(\underline{U}), S\Lambda^s U) &= (\Lambda^s(\varepsilon\nabla b\cdot \underline{v}), \Lambda^s\zeta) + (\Lambda^s(\underline{\mathfrak{T}}^{-1} r_2(\underline{U})), \underline{\mathfrak{T}}\Lambda^s v)\\ &= (\Lambda^s(\varepsilon\nabla b\cdot \underline{v}), \Lambda^s\zeta) - \varepsilon\mu\big([\Lambda^s, \underline{\mathfrak{T}}]\underline{\mathfrak{T}}^{-1} r_2(\underline{U}), \Lambda^s v\big)\\ &\quad +\varepsilon\mu\big(\Lambda^s r_2(\underline{U}), \Lambda^s v\big).\end{aligned}$$

Using again here the explicit expressions of $\underline{\mathfrak{T}}$, $r_2(\underline{U})$ and Lemma 2, we get

$$(\Lambda^s B(\underline{U}), S\Lambda^s U) \le \varepsilon C(E^s(\underline{U}))E^s(U).$$

• Estimate of $(\Lambda^s v, [\partial_t, \underline{\mathfrak{T}}]\Lambda^s v)$. We have that

$$\begin{aligned}(\Lambda^s v, [\partial_t, \underline{\mathfrak{T}}]\Lambda^s v) &= (\Lambda^s v, \partial_t \underline{h}\Lambda^s v) + \frac{\mu}{3}(\Lambda^s\nabla\cdot v, \partial_t \underline{h}^3\Lambda^s\nabla\cdot v)\\ &\quad -\frac{\varepsilon\mu}{2}(\Lambda^s v, \partial_t\underline{h}^2\nabla b\Lambda^s\nabla\cdot v) - \frac{\varepsilon\mu}{2}(\Lambda^s\nabla\cdot v, \partial_t\underline{h}^2\nabla b\cdot\Lambda^s v)\\ &\quad +\varepsilon^2\mu(\Lambda^s v, \partial_t\underline{h}\nabla b\nabla b^T\Lambda^s v).\end{aligned}$$

Controlling these terms by $\varepsilon C(E^s(\underline{U}), |\partial_t\underline{h}|_{L^\infty})E^s(U)^2$ follows directly from a Cauchy-Schwarz inequality and an integration by parts.

Gathering the informations provided by the above estimates and using the fact that the embedding $H^s(\mathbb{R}^2)\subset W^{1,\infty}(\mathbb{R}^2)$ is continuous, we get

$$e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(U)^2) \le \varepsilon\big(C(E^s(\underline{U}), |\partial_t\underline{h}|_{L^\infty}) - \lambda\big)E^s(U)^2 + \varepsilon C(E^s(\underline{U}))E^s(U).$$

Taking $\lambda = \lambda_T$ large enough (how large depending on $\sup_{t\in[0,\frac{T}{\varepsilon}]} C(E^s(\underline{U}), |\partial_t\underline{h}|_{L^\infty})$ to have the first term of the right hand side negative for all $t\in[0,\frac{T}{\varepsilon}]$, one deduces

$$\forall t\in[0,\frac{T}{\varepsilon}],\qquad e^{\varepsilon\lambda t}\partial_t(e^{-\varepsilon\lambda t}E^s(U)^2) \le \varepsilon C(E^s(\underline{U}))E^s(U).$$

Integrating this differential inequality yields therefore

$$\forall t\in[0,\frac{T}{\varepsilon}],\qquad E^s(U) \le e^{\varepsilon\lambda_T t}E^s(U_0) + \varepsilon\int_0^t e^{\varepsilon\lambda_T(t-t')}C(E^s(\underline{U})(t'))dt',$$

which is the desired result. □

3.3. **Main result.** In this subsection we prove the main result of this paper, which shows the well-posedness of the new Green-Naghdi equations (6) for times of order $O(\frac{1}{\varepsilon})$.

Theorem 1. *Let $b\in C_b^\infty(\mathbb{R}^2)$, $t_0>1$, $s\ge t_0+1$. Let also $U_0 = (\zeta_0, v_0^T)^T\in X^s$ be such that (16) is satisfied. Then there exists a maximal $T_{max}>0$, uniformly bounded from below with respect to $\varepsilon,\mu\in(0,1)$, such that the new Green-Naghdi equations (6) admit a unique solution $U = (\zeta, v^T)^T\in X^s_{T_{max}}$ with the initial condition $(\zeta_0, v_0^T)^T$ and preserving the nonvanishing depth condition (16) for any $t\in[0,\frac{T_{max}}{\varepsilon})$. In particular if $T_{max}<\infty$ one has*

$$|U(t,\cdot)|_{X^s}\longrightarrow\infty \quad as \quad t\longrightarrow\frac{T_{max}}{\varepsilon},$$

or

$$\inf_{\mathbb{R}^2} h(t,\cdot) = \inf_{\mathbb{R}^2} 1+\varepsilon(\zeta(t,\cdot)-b(\cdot))\longrightarrow 0 \quad as \quad t\longrightarrow\frac{T_{max}}{\varepsilon}.$$

Remark 5. *For 2D surface waves, non flat bottoms, B. A. Samaniego and D. Lannes* [3] *proved a well-posedness result for the standard 2D Green-Naghdi equations using a Nash-Moser scheme. Our result only uses a standard Picard iterative and there is therefore no loss of regularity of the solution of the new 2D Green-Naghdi equations with respect to the initial condition.*

Remark 6. *No smallness assumption on ε nor μ is required in the theorem. The fact that T_{max} is uniformly bounded from below with respect to these parameters allows us to say that if some smallness assumption is made on ε, then the existence time becomes larger, namely of order $O(1/\varepsilon)$.*

Proof. Using the energy estimate of Proposition 1, one proves the result following the same lines as in the proof of Theorem 1 in [7], which is itself an adaptation of the standard proof of well-posedness of hyperbolic systems (e.g [1, 17]). □

3.4. **Conservation of the almost irrotationality of v.** To obtain the new $2D$ Green-Naghdi model (6) we used the fact that $\mathrm{curl}\, v = O(\mu)$, we prove in the following Theorem that the new model (6) conserves of course this property.

Theorem 2. *Let $t_0 > 1$, $s \geq t_0 + 1$, $U_0 = (\zeta_0, v_0^T)^T \in X^{s+4}$ with $|\mathrm{curl}\, v_0|_{H^s} \leq \mu C(|U_0|_{X^s})$. Then, the solution $U = (\zeta, v^T)^T \in X^{s+4}_{T_{max}}$ of the new Green-Naghdi equations (6) with the initial condition $(\zeta_0, v_0^T)^T$ satisfies*

$$\forall\, 0 < T < T_{max}, \quad |\mathrm{curl}\, v|_{L^\infty([0,\frac{T}{\varepsilon}],H^s)} \leq \mu\, C(T, |U_0|_{X^{s+4}}).$$

Proof. Applying the operator $\mathrm{curl}\,(\cdot)$ to the second equation of the model (6) after multiplying it by $\dfrac{1}{h}$ yields

$$\partial_t w + \varepsilon \nabla \cdot (vw) = \varepsilon\mu f_1 + g,$$

where

$$\begin{aligned} f_1 &= -\mathrm{curl}\,\Big(\frac{1}{h}\mathcal{T}[h,\varepsilon b] - \frac{1}{h}\nabla^\perp \mathrm{curl}\,\Big) v \cdot \nabla v \\ &\quad -\mathrm{curl}\,\frac{1}{h}\Big\{\frac{2}{3}\nabla[h^3(\partial_1 v \cdot \partial_2 v^\perp + (\nabla\cdot v)^2)] + \Re[h,\varepsilon b](v)\Big\} \end{aligned}$$

and

$$g = -\mu \mathrm{curl}\,\Big(\frac{1}{h}\mathcal{T}[h,\varepsilon b] - \frac{1}{h}\nabla^\perp \mathrm{curl}\,\Big)\partial_t v.$$

From the identity $\mathrm{curl}(\frac{1}{h}W) = -\varepsilon\frac{1}{h^2}\nabla^\perp(\zeta - b)\cdot W + \frac{1}{h}\mathrm{curl}\, W$, we deduce that g can be put under the form

$$g = \varepsilon\mu f_2 + \mu \mathrm{curl}\,\big(\frac{1}{h}\nabla^\perp \partial_t w\big),$$

with

$$f_2 = \frac{1}{h^2}\nabla^\perp(\zeta - b)\cdot \mathcal{T}[h,\varepsilon b]\partial_t v.$$

We have thus shown that w solves

$$\Big(I - \mu \mathrm{curl}\,\big(\frac{1}{h}\nabla^\perp\big)\Big)\partial_t w + \varepsilon\nabla\cdot(vw) = \varepsilon\mu f_1 + \varepsilon\mu f_2.$$

Energy estimates on this equation then show that for all $0 < T < T_{max}$, one has

$$|w|_{L^\infty([0,T/\varepsilon],H^s)} \leq \mu C(T, |U|_{X^s_T})\big(|f_1|_{L^\infty([0,T/\varepsilon],H^s)} + |f_2|_{L^\infty([0,T/\varepsilon],H^s)}\big).$$

Now, one deduces from the explicit expression of f_j $(j=1,2)$ that $|f_j|_{L^\infty([0,T/\varepsilon],H^s)} \leq C(|U|_{X_T^{s+4}})$; with Theorem 1, we deduce that $|f_j|_{L^\infty([0,T/\varepsilon],H^s)} \leq C(T,|U_0|_{X_T^{s+4}})$, and the result follows. $\square$

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